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A MULTITEMPERATURE BOUNDARY LAYER

by

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ABSTRACT

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This is a preliminary report on an analytical investigation of the end wall boundary layer behind a reflected shock in a shock tube in highly ionized argon. Two nonequilibrium effects are considered: chemical nonequilibrium because of finite recombination rates, and lack of equilibration between electron and heavy (atom and ion) temperatures because of a finite rate of energy exchange due to elastic collisions between electrons and ions. The Debye length is small compared to the boundary layer thickness. In contrast with previous work in this area, we retain the coupling between the electrical effects and the aerodynamic effects, considering both simultaneously. The electrical effects principally manifest themselves in a plasma sheath next to the wall which is a few Debye lengths thick. In this sheath the simple molecular description of the plasma originated by Langmuir and Mott-Smith is used. Outside the sheath, in the aerodynamic boundary layer, the usual continuum fluid mechanics treatment is used. These two descriptions of the plasma are joined at the sheath edge by requiring continuity of ion mass flux and electron energy flux, which serve as boundary conditions for the continuum boundary layer equations. The basic information determining the chemical and transport properties of ionized argon are well-known, and they are used to develop a set of properties for the two-temperature case considered here. This completes the information necessary to fully define the problem. The solution is attempted by means of a local similarity approach, in which time is treated as a parameter, and the boundary layer equations are reduced to ordinary differential equations. Only a few solutions, all for frozen chemistry, are available at this stage of the investigation. They show that when elastic energy exchange between ions and electrons is ignored, the electron temperature at the sheath edge is about half the external temperature. However, this energy exchange has only a slight effect on wall heat transfer rate. Further calculations for both frozen chemistry and finite recombination rate cases will be pursued.

H. A. H. H.

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I. INTRODUCTION

There are a considerable number of devices where a hot ionized plasma comes in contact with a cold wall. Examples are plasma arcs, plasma propulsion motors, stagnation point flow in high speed flight, and many fusion machines. Theoretical attempts to calculate the heat transfer to the walls usually separate the electrical properties from the aerodynamic flow.^{1, 2, 3, 4} However, there are many situations where the degree of ionization of the plasma is large and this separation is not valid. In addition, the state of the plasma in the boundary layer is usually assumed to be either chemically frozen (no gas phase recombination) or in complete chemical equilibrium. These limiting assumptions permit the solution of the boundary layer equations by the usual similarity transformations. However, these limiting cases need not bracket the general case which must include recombination chemistry and different electron and heavy particle temperatures. In particular, the electron temperature can be considerably above the wall temperature even near the wall.

The main objective of this paper is the development of a general boundary layer theory which will include the important electrical and chemical effects directly in the boundary layer equations.¹⁰ To obtain specific results, we have considered a Rayleigh problem in which a hot plasma at rest is placed in contact with a cold wall. Detailed calculations are presented for argon, since the argon recombination rates,^{5, 6} and the appropriate transport properties^{7, 8} are well known. In addition, these calculations can be compared directly to experimental data in which the degree of ionization is up to 50%.⁹ For this case, the electrical effects are important in the determination of the boundary layer profile. For example, it would be in error to couple the electrical effects to the boundary layer profile for the neutral component as is done for low degrees of ionization. A quantitative understanding of the following phenomena is needed in order to obtain solutions of the boundary layer equations: (1) electron and ion temperature equilibration rates; (2) argon recombination chemistry; (3) plasma sheath potential; (4) thermal conductivity of the atoms, ions, and electrons; and ambipolar diffusion of electron-ion pairs through atoms.

These phenomena will be discussed in the next section. Then the boundary layer equations will be developed. Finally, we will present the solutions of the boundary layer equations that have been obtained to date.

II. ARGON PLASMA PROPERTIES

The end wall boundary layer using the gas sample behind the reflected shock in a shock tube is essentially a Rayleigh problem. A hot plasma sample is abruptly placed in contact with a cold wall. There is only a thermal boundary layer, whose properties depend on the concentrations, thermal conductivity, and diffusion rates of each species. This permits a simplification of the boundary layer theory. Effects of viscosity do not appear, since there is no flow parallel to the wall.

The evaluation of the end wall heat transfer rates requires a knowledge of the chemical species concentration in the boundary layer together with the transport properties that are important in the theory. These are the thermal conductivities of the electrons and neutral argon, and the ambipolar diffusion of electron-ion pairs through neutral argon. In addition, the slow rate of energy exchange between electrons and ions by elastic collisions permits different electron and ion temperatures. The boundary conditions at the wall must be modified because of the plasma sheath potential, which effectively isolates the electrons from the wall. We will now consider these phenomena in more detail.

Energy Exchange by Elastic Collisions

The exchange of energy between particles in an elastic collision is a function of the ratio of the mass of the particles. For particles of equal mass, the temperature equilibrates in a few collisions, as is the case for collisions between the argon atom and ion. However, it takes roughly 10^5 elastic collisions for electrons and argon atoms or ions to equilibrate their temperature, because the electron is 80,000 times lighter. This large number of collisions results in a weaker coupling between the electrons and heavy component, thus permitting the electron and heavy particles separately to have distinct but different temperatures. In particular, for the argon boundary layer, the electron temperature remains higher than that for the heavy particles near the wall. This difference is due mainly to three effects: (1) the large number of collisions required for equilibration of electron and heavy particle temperature; (2) ionic recombination heats only the electron component; and (3) the electron component is thermally insulated from the wall by the plasma sheath.

The rate of energy transfer per unit volume by elastic collisions E_{el} from electrons to single charged ions is⁵

$$E_{el} = \frac{n_E^2 \epsilon^4}{m_A} \sqrt{\frac{8\pi m_e}{kT_E}} \left(\frac{T_H}{T_E} - 1 \right) \ln \frac{9(kT_E)^3}{8\pi n_E \epsilon^6}. \quad (2.1)$$

n_E is the electron number density per cm^3 , and m_E , m_H , T_E and T_H are the electron and ion masses and temperatures, respectively. e and k are the electric charge and Boltzman's constant.

The energy transfer by elastic collisions from electrons to atoms is usually much less than to ions. The atom-electron elastic collision cross section is small for argon because of the strong Ramsauer effect.¹¹ The cross section has a minimum of $5 \times 10^{-17} \text{ cm}^2/\text{atom}$ in the range from 2000°K to $10,000^\circ\text{K}$. The ion-electron elastic cross section predominates over that for the electron-atom when the degree of ionization is in excess of 10^{-3} . Therefore, we have not included the atom-electron energy transfer in the present analysis.

Argon Recombination Kinetics

The only important recombination reaction for argon is by the electron impact process



where E , Ar^+ and Ar designate electrons, ionized and neutral argon, respectively. (Radiative recombination is negligible because of the trapping of the resonance radiation by the large neutral atom concentration in the boundary layer. Also negligible is the two step process with the molecule Ar_2^+ in the intermediate stage because of the small binding energy of Ar_2^+ .) The rate of electron production is governed by the relation

$$\frac{dn_E}{dt} = -k_{\text{rec}} n_E^2 n_I + k_{\text{ion}} n_E n_A , \quad (2.3)$$

where n_E , n_I and n_A are the number densities of electrons, ions and atoms, and k_{rec} and k_{ion} are the recombination and ionization rate constants.

At thermodynamic equilibrium $dn_E/dt = 0$, and the rate constants of Eq. (2.3) are proportional to the equilibrium constant, K_{eq} :

$$\frac{k_{\text{ion}}}{k_{\text{rec}}} = \left(\frac{n_E n_I}{n_A} \right)_{\text{eq}} = K_{\text{eq}} , \quad (2.4)$$

where the subscript eq indicates equilibrium. K_{eq} is obtained from the Saha formula applied to argon: $K_{\text{eq}} = 2.9 \times 10^{16} T_E^{1.5} \exp(-T_{\text{ion}}/T_E)/\text{cm}^3$.

The argon recombination rate constant can be obtained from two sources: (1) the measured ionization rates at high temperatures,⁵ and (2) the ionic recombination theory^{12, 13, 14} based on the classical electron impact cross sections.¹⁵ The recombination rate constant, $k_{\text{rec}}^{\text{cl}}$, based on classical theory has the simple relation¹⁴

$$k_{\text{rec}}^{\text{cl}} = 1.16 \times 10^{-8} T_E^{-4.5} \text{ cm}^6/\text{electron}^2\text{-sec.} \quad (2.5)$$

This result, based on classical theory, is not expected to apply above 4000°K, where the important transition in the recombination process is the large quantum jump from the first excited state to the ground state.¹³

The ionization process for electron temperatures in excess of 10,000°K was shown by Petschek and Byron⁵ to be determined only by the rate of excitation of the lower excited states, say k_{exc} , the rates for subsequent ionization from the low excited states being much faster. Thus, for

$$T_E > 10,000^\circ K, \quad k_{ion} = k_{exc}. \quad (2.6)$$

k_{exc} , the excitation rate constant to the lowest excited states is⁵

$$k_{exc} = 4.4 \times 10^{-6} \left[\frac{2(KT_E)^3}{m_E} \right]^{1/2} \left(2 + \frac{T_{exc}}{T_E} \right) e^{-T_{exc}/T_E} \frac{cm^3}{electron-sec}. \quad (2.7)$$

$T_{exc} = 134,000^\circ K$ is the temperature corresponding to the energy of the first excited state. For $T_E > 10,000^\circ K$, the recombination rate constant can be determined from k_{exc} by combining Eqs. (2.4), (2.6) and (2.7)

$$k_{rec}^{PB} = \frac{k_{exc}}{K_{eq}} = (8.9 \times 10^{-28}/T_E) e^{48,000/T_E} \frac{cm^6}{electron^2-sec}. \quad (2.8)$$

$T_{ion} - T_{exc} = 48,000^\circ K$ corresponds to the ionization potential of the first excited level. This rate constant requires the assumption that the population of each excited argon level is in equilibrium with the electrons; the loss rate from any level is much slower than the production rate. The rate limiting process becomes the de-activation of the first excitation level by the electron collision process. (Radiative transitions do not contribute, because ground state resonance radiation is trapped at high densities.) The true rate constant, k_{rec} , cannot be larger than either of the two rates given by Eqs. (2.5) and (2.8), and we propose that k_{rec} is given by Eq. (2.5) for $T_E < 3000^\circ K$ and by Eq. (2.8) for $T_E > 3000^\circ K$.

The large change of the recombination rate constant with temperature has an important effect on the boundary layer chemistry for the case of large degrees of ionization: that is, for electron concentrations from 10^{15} to 10^{18} , and times from 1 to 10 microseconds. For electron temperatures above about 10,000°K, the recombination rate is small, and there is negligible recombination. However, for lower electron temperatures, the recombination rate becomes so fast that the degree of ionization must remain in equilibrium with the local electron temperature; that is, the degree of ionization is determined by the local electron temperature through the Saha equation.

Wall Sheath Properties

The properties of the plasma sheath surrounding an object in contact

with a plasma have been extensively treated.^{1, 2, 3, 16, 17} An approximate theory has been postulated^{16, 17} in which the plasma has a continuum description with charged neutrality up to a sheath edge. Charge neutrality breaks down in the sheath, an electric potential develops between the plasma and the wall, and a molecular description of the plasma is used. The sheath thickness is of the order of a Debye length, h :

$$h = \sqrt{\frac{kT_E}{4\pi n_E e^2}} = 6.90 \sqrt{\frac{T_E}{n_E}} \text{ cm}, \quad (2.9)$$

where T_E is in $^{\circ}\text{K}$ and n_E is the number of electrons per cm^3 .

The sheath potential adjusts itself to maintain charge neutrality at the edge of the plasma by not allowing a net charge to cross the sheath. That is, the electron current towards the wall equals the total positive current. The positive current consists of two components: ions diffusing toward the surface and electron emission from the surface. Only electrically conducting walls will be treated, and these walls are catalytic. Ion-electron pairs will neutralize on contact with the wall, and leave as atoms. The sheath potential, ϕ , is obtained from the formula

$$\frac{n_E \bar{c}_E \exp(-|e\phi|/kT_E)}{4} = \frac{n_I c_I}{4} + J, \quad (2.10)$$

where n_I and n_E are the ion and electron number density at the edge of the sheath; \bar{c}_E is the mean thermal velocity for the electrons, and J is the wall electron emission. Note in this approximation, $n_I = n_E$ at the edge of the sheath. c_I is determined by the drift velocity of the ion in the sheath; and reduces to the ion thermal velocity in equilibrium with the local atom temperature, when the ion makes many collisions in passing through the sheath. When the ions fall freely through the sheath without collisions,¹⁷ $c_I = 4\sqrt{kT_E/m_I}$. The wall emission, J , can be produced by field emission and by photoelectric emission from ultraviolet light irradiation. It can be shown that the photoelectric effect is negligible for argon. The amount of current leaving the wall due to field emission is difficult to estimate, since the field strengths at the wall, based on the sheath thickness of the order of 1000 \AA , can be in excess of 10^5 volts/cm. However, if one assumes negligible wall electron emission, the sheath potential in Eq. (2.10) reduces to

$$\phi = \frac{kT_E}{e} \ln \frac{\bar{c}_E}{c_I}. \quad (2.11)$$

The sheath potential becomes a function of T_E and c_I . We anticipate that the electron temperature at the edge of the plasma sheath will be a substantial fraction of the external temperature, while the ions and atoms are at the wall temperature of approximately 300°K . When the free stream is 50% ionized, the electron-ion pair and neutral atom number densities at the

edge of the sheath are the order of $10^{17}/\text{cm}^3$ and $10^{20}/\text{cm}^3$. With these conditions, the sheath thickness, h is about 10^{-6}cm . To determine the effective ion velocity, c_I , the sheath thickness is compared to the mean free path for atom-ion collisions, which is $5 \times 10^{-7}\text{cm}$. The ion makes about 2 collisions in passing through the sheath. For this example, the value of c_I to be used in Eq. (2.11) is intermediate between the values discussed above. The energy that the ion receives from the sheath voltage is transferred to the atoms. However, the sheath is so small compared to the boundary layer thickness that this energy is immediately transferred to the wall by the atoms.

The energy transported across the sheath potential by the electrons and ions was treated by Jukes.¹⁰ On the average, each electron crossing the sheath reaches the wall with an energy equal to $2\kappa T_E$, while an ion carries the thermal energy $2\kappa T_H$ plus the sheath potential energy, $|\epsilon\phi|$. In addition, ion and electron pair recombined at the surface releasing the ionization energy, E_I per pair. Thus, each ion-electron pair brings to the wall the net energy, $2\kappa T_E + E_I + |\epsilon\phi|$. When $T_E = 300^\circ\text{K}$, then the only important term is the ionization energy, E_I . However, for an electron temperature of $10,000^\circ\text{K}$ at the sheath, the total energy brought to the wall is 24 eV/pair.

Several authors^{1, 2, 3} have calculated the electrical properties of a plasma without using the simple sheath description outlined above. However, they have only considered small degrees of ionization, and taken the aerodynamic properties of the plasma as given, thus uncoupling the aerodynamic and electric effects. For the large degrees of ionization and large plasma densities of interest here, such an uncoupling is not possible. However, the Debye thickness is small compared to the boundary layer thickness under these conditions, so the approximate description of the electrical effects given by the sheath model outlined above is sufficiently accurate.

Transport Properties

The transport properties of argon needed for the end wall boundary layer calculation are the thermal conductivity of the atoms, ions, and electrons, and the ambipolar diffusion coefficient for ion-electron pairs diffusing through atoms.

For the one-temperature boundary layer, these properties for argon are fairly well-known. The derivation of a set of properties for the two-temperature case will be based on the one-temperature set given in Ref. 9. The changes necessary in the thermal conductivity can be found by going back to the source of the mixture rule used in Ref. 9, which is found in Ref. 7. A reconsideration of the derivation leads to only a slight modification in one term, since all other appearances of the ratio T_H/T_E are in connection with terms of the form $(m_E/m_A)^{1/2}$ which are ignored. The result is that the contribution of ions to the mixture thermal conductivity is still negligible, the contribution of the electrons is

$$k_E = k_s \left[1 + \sqrt{2} \frac{1-\alpha}{\alpha} \frac{k_s}{k_A} \frac{Q_{EA}}{Q_{AA}} \sqrt{\frac{m_E T_H}{m_A T_E}} \right]^{-1}, \quad (2.12a)$$

and that of the atoms is

$$k_A = k_A^* \left[1 + \frac{\alpha}{1-\alpha} \frac{Q_{AI}}{Q_{AA}} \right]^{-1}. \quad (2.12b)$$

Here Q is the collision cross section between particles denoted by the subscripts, α is the fraction ionized, T_H the heavy temperature, and k_s is the Spitzer conductivity, which is taken to be

$$k_s = \frac{4.4 \times 10^{-13} T_E^{5/2}}{\frac{1}{4} \ln[\Lambda_1^4 + \Lambda_2^4 + e^4]} \frac{\text{cal}}{\text{cm-sec-}^\circ\text{K}}. \quad (2.13)$$

The Λ quantities are (T_E in $^\circ\text{K}$, n_E in particles/cm³)

$$\Lambda_1 = 1.24 \times 10^{14} T_E^{3/2} n_E^{-1/2}, \quad \Lambda_2 = 1800 T_E n_E^{-1/3}, \quad (2.14)$$

which are the ratio of the Debye distance to the impact parameter for 90° deflection, and the ratio of the average inter-electron distance to the impact parameter for 90° deflection, respectively. The combination used in k_s allows for a proper crossover between Λ_1 , valid at low electron density and high temperature, and Λ_2 , valid in the other limit. The pure atom conductivity is taken to be

$$k_A^* = 5.8 \times 10^{-7} T_H^{3/4} \text{ cal/cm-sec-}^\circ\text{K}. \quad (2.15)$$

The ratio Q_{AE}/Q_{AA} is relatively unimportant in determining k_E since this term is small except for α very near zero. So a constant value of 1.5×10^{-2} was used, in accordance with average values of these cross sections is the temperature region of interest. The ratio Q_{AI}/Q_{AA} is the same as that derived in Ref. 9 and is

$$Q_{AI}/Q_{AA} = 1.44 T_H^{0.16}. \quad (2.16)$$

The ambipolar mass flux of ions is found by using a momentum equation for atoms diffusing in a two component mixture of atoms and ion-electron pairs. The derivation follows that of Ref. 9 to the point where the flux of ion-electron pairs is expressed as

$$n_I m_I V_I = \frac{n_A D_{IA} p}{k T_H} \nabla \left(\frac{p_A}{p} \right), \quad (2.17)$$

where D_{IA} is the binary ion-atom diffusion coefficient, p_A the atom partial pressure, and the mixture pressure, p , has been taken constant, as appropriate to diffusion in a boundary layer. The departure from a one-temperature case occurs in the evaluation of p_A/p . For the present case

$$p_A = n_A k T_H, \quad p = (n_A + n_I) k T_H + n_E k T_E, \quad (2.18)$$

$$p_A/p = (1-\alpha)/(1+\alpha T_E/T_H),$$

where the fraction ionized, α , is expressible in terms of the number densities by

$$n_E = n_I = \alpha (n_A + n_I). \quad (2.19)$$

We thus find

$$n_I m_I V_I = \frac{-p D_{IA}}{1+\alpha T_E/T_H} \left[\left(1 + \alpha \frac{T_E}{T_H} \right) \nabla \alpha + \alpha (1-\alpha) \nabla \frac{T_E}{T_H} \right]. \quad (2.20)$$

For $T_E = T_H$, this reduces to the value used in Ref. 9. When the temperatures are not equal, there are two new effects. One is the dependence of the coefficients on T_E/T_H , the other the appearance of an effect of the gradient of the temperature ratio. For small α , the latter effect disappears and the two temperatures just replace the usual ambipolar factor 2 by $(1 + T_E/T_H)$.

To normalize the diffusion coefficient, a Lewis number is defined, as in Ref. 9, by

$$L_A \equiv \frac{2 D_{IA}}{1+\alpha} \frac{\rho c_p}{h_{2A}}, \quad (2.21)$$

which is shown there to be

$$L_A = \frac{8}{5} \frac{Q_{AA}}{Q_{AI}} = 1.11 T_H^{-0.16}. \quad (2.22)$$

If this is used, the ion mass diffusion flux can be written:

$$n_I m_I \mathbf{V}_{-I} = - \frac{L_A \beta_A}{c_p} \frac{(1 + T_E/T_H) \nabla \alpha + \alpha(1 - \alpha) \nabla (T_E/T_H)}{2(1 + \alpha T_E/T_H)(1 + \alpha)^{-1}} \quad (2.23)$$

III. BOUNDARY LAYER FORMULATION

General Equations

Since we are allowing the electrons to have a temperature different from the heavy particles (atoms and ions), we must return to fundamentals to obtain the equations of motion. A kinetic theory approach can be used, leading to slight modifications of the equations in Chapter 8 of Chapman and Cowling.¹⁸

Allowing for a production term w_i (the mass rate of production of species i per unit volume and time), the mass conservation law for this species is

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot \rho_i (\underline{v} + \underline{V}_i) = w_i, \quad (3.1)$$

where \underline{v} is the mass velocity, \underline{V}_i the diffusion velocity of species i relative to \underline{v} and ρ_i the partial density $\rho_i = n_i m_i$. When this is summed over all species, we find the usual over-all continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0, \quad (3.2)$$

where

$$\sum \rho_i \equiv \rho, \quad \sum \rho_i \underline{V}_i = 0, \quad \sum w_i = 0. \quad (3.3)$$

For the momentum conservation law for species i , we introduce the external force per unit mass \underline{F}_i and the shear stress tensor $\underline{\tau}_i$, defined by

$$\underline{p}_i = \rho_i \underline{U} + \underline{\tau}_i, \quad (3.4)$$

where \underline{p}_i is the stress tensor for species i , and \underline{U} the unit tensor. Then the momentum equation can be written

$$\rho_i \frac{D\mathbf{v}}{Dt} + \nabla p_i + \nabla \cdot \underline{\underline{\tau}}_i - \rho_i \underline{\underline{F}}_i + \frac{D\rho_i \underline{\underline{V}}_i}{Dt} + \rho_i \underline{\underline{V}}_i \nabla \cdot \mathbf{v} + (\rho_i \underline{\underline{V}}_i \cdot \nabla) \mathbf{v} = \underline{\underline{\Delta M}}_i, \quad (3.5)$$

where $\underline{\underline{\Delta M}}_i$ is the increase of momentum of species i due to collisions. All the terms on the second line vanish when a sum over all species is performed, leading to the over-all momentum equation

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p + \nabla \cdot \underline{\underline{\tau}} - \sum \rho_i \underline{\underline{F}}_i = 0, \quad (3.6)$$

where

$$\sum \rho_i = \rho, \quad \sum \underline{\underline{\tau}}_i = \underline{\underline{\tau}}. \quad (3.7)$$

Finally, the energy conservation law is written using the thermal enthalpy per unit mass, h_i , and the energy flux vector $\underline{\underline{q}}_i$:

$$\frac{D\rho_i h_i}{Dt} + \rho_i h_i \nabla \cdot \mathbf{v} + \nabla \cdot \underline{\underline{q}}_i - \frac{D\rho_i}{Dt} + \underline{\underline{\tau}}_i : \nabla \mathbf{v} - \rho_i \underline{\underline{V}}_i \cdot \underline{\underline{F}}_i + \rho_i \underline{\underline{V}}_i \cdot \frac{D\mathbf{v}}{Dt} = \underline{\underline{\Delta E}}_i. \quad (3.8)$$

Here $\underline{\underline{\Delta E}}_i$ is the energy gained by species i due to collisions, including chemical reaction effects. When this equation is summed over all species, the last term on the left vanishes, but the collision term might still make a contribution because of conversion of kinetic energy into potential energy by chemical reaction. The result is the over-all energy equation

$$\frac{D\rho h}{Dt} + \rho h \nabla \cdot \mathbf{v} + \nabla \cdot \underline{\underline{q}} - \frac{D\rho}{Dt} + \underline{\underline{\tau}} : \nabla \mathbf{v} - \sum \rho_i \underline{\underline{V}}_i \cdot \underline{\underline{F}}_i = \underline{\underline{\Delta E}}, \quad (3.9)$$

where

$$\sum p_i h_i \equiv p h, \quad \sum q_i \equiv q, \quad \sum \Delta E_i = \Delta E. \quad (3.10)$$

For a mixture of monatomic gases, the thermodynamic expressions are

$$p_i = \rho_i K T_i / m_i, \quad h_i = c_{pi} T_i, \quad c_{pi} = 5K/2m_i. \quad (3.11)$$

For the energy flux vector, we shall take the usual expression

$$q_i = -k_i \nabla T_i + p_i \underline{V}_i h_i, \quad (3.12)$$

thus ignoring possible diffusive thermo-effects, which may be different in a multi-temperature mixture than in a single temperature mixture. Note that the thermal conduction coefficient k_i is the conductivity of species i in the mixture, not by itself. We shall have no need for an expression for the shear stress tensor $\underline{\tau}_i$.

To apply these equations to the present case of a monatomic, ionized gas, we first apply Eq. (3.5) to the momentum of the electrons. All except three terms have the electron mass m_E as a factor. The stress term depends on $m_E^{1/2}$. The electron pressure gradient term, and the external force term are independent of m_E , since $\underline{F}_E = -e\underline{E}/m_E$, where e is the electronic charge and \underline{E} the electric field. Thus, if we ignore factors of $m_E^{1/2}$ and smaller, the electron momentum equation becomes

$$\nabla p_E = -\rho_E e \underline{E} / m_E = -\rho_I e \underline{E} / m_I, \quad (3.13)$$

which relates the electric field between electrons and ions to the electron pressure.

The over-all momentum equation in this case is

$$\rho \frac{D\underline{v}}{Dt} + \nabla p + \nabla \cdot \underline{\tau} = 0, \quad (3.14)$$

because the external force term balances out between electrons and ions.

When the energy equation (3.8) is written for the electrons, the

last term on the left is proportional to m_E and can be ignored, while the shear stress term depends on m_E^2 and can also be dropped. The collision term contains two contributions. One is the loss of energy by elastic collisions between electrons and the heavy particles, discussed in Section II, which we denote by E_{el} . The second contribution is the loss of energy to the ions in the ionization reaction, which is $w_I h^0$, where h^0 is the ionization energy per unit mass of the ions, E_I/m_A . When we use Eq. (3.13) for the external force term and the assumption of ambipolar diffusion, $\underline{V}_E = \underline{V}_I$, the electron energy equation is

$$\begin{aligned} \frac{D \rho_E h_E}{Dt} + \rho_E h_E \nabla \cdot \underline{v} + \nabla \cdot \underline{q}_E \\ - \frac{D \rho_E}{Dt} - \underline{V}_I \cdot \nabla \rho_E = -w_I h^0 - C_{el} E_{el}. \end{aligned} \quad (3.15)$$

C_{el} is a rate constant for the elastic energy exchange process inserted for convenience in the numerical calculations. The physically correct value of this constant is unity.

Since the ions and atoms have the same temperature, we will combine their energy equations. When we do, the last term on the left disappears since $\sum \rho_i \underline{V}_i = 0$ and only the heavies contribute to this sum. The collision term involves only the increase of energy of the heavies due to elastic collisions. So the heavy energy equation is

$$\begin{aligned} \frac{D}{Dt} (\rho_I h_I + \rho_A h_A) + (\rho_I h_I + \rho_A h_A) \nabla \cdot \underline{v} + \nabla \cdot (\underline{q}_I + \underline{q}_A) \\ - \frac{D}{Dt} (\rho_I + \rho_A) + \underline{V}_I \cdot \nabla \rho_E + (\underline{\tau}_I + \underline{\tau}_A) : \nabla \underline{v} = C_{el} E_{el}, \end{aligned} \quad (3.16)$$

where again Eq. (3.13) has been used for the external force term. The over-all energy equation is the sum of Eqs. (3.15) and (3.16):

$$\begin{aligned} \frac{D \rho h}{Dt} + \rho h \nabla \cdot \underline{v} + \nabla \cdot (\underline{q}_I + \underline{q}_A + \underline{q}_E) - \frac{D \rho}{Dt} \\ + (\underline{\tau}_I + \underline{\tau}_A) : \nabla \underline{v} = -w_I h^0. \end{aligned} \quad (3.17)$$

End Wall Boundary Layer Equations

We now apply the general equations to the end wall boundary layer behind a reflected shock, i.e., with a quiescent external flow. Taking the y axis normal to the wall, variations occur only in y and t. There is no flow parallel to the wall, so the tangential momentum equations are of no significance. Applying the usual boundary layer approximations, the over-all momentum equation (3.16) in the normal direction yields the familiar result that the mixture pressure p is a constant throughout the layer.

The remaining equations are the over-all mass conservation equation (3.2) which determines the normal velocity v; the species conservation equations (3.1), which reduce to one equation for the fraction dissociated by using the ion equation; and the electron, heavy and over-all energy equations, (3.15), (3.16), and (3.17), only two of which are required to determine the electron and heavy temperatures T_E and T_H . When the boundary layer approximations are used in these equations, the shear stress terms all disappear. We also make use of the thermodynamic relations (3.11), the energy flux expression (3.12), and the diffusion mass flux sum of Eq. (3.3). The equations of the end wall boundary layer then become

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y} \rho v = 0, \quad (3.18a)$$

$$\rho \frac{D_y}{Dt} \alpha + \frac{\partial}{\partial y} \rho_I V_I = w_I \quad (3.18b)$$

$$\rho \frac{D_y}{Dt} (\alpha c_{pH} T_E) - \frac{D_y p_E}{Dt} - V_I \frac{\partial p_E}{\partial y} \quad (3.18c)$$

$$+ \frac{\partial}{\partial y} (-k_E \frac{\partial T_E}{\partial y} + \rho_I V_I c_{pH} T_E) = -w_I h^0 - C_{el} E_{el},$$

$$\begin{aligned} \rho \frac{D_y}{Dt} (c_{pH} T_H) + \frac{D_y p_E}{Dt} + V_I \frac{\partial p_E}{\partial y} \\ + \frac{\partial}{\partial y} (-k_H \frac{\partial T_H}{\partial y}) = C_{el} E_{el}, \end{aligned} \quad (3.18d)$$

$$\rho \frac{D_y}{Dt} [c_{PH}(T_H + \alpha T_E)] \quad (3.18e)$$

$$+ \frac{\partial}{\partial y} \left(-k_H \frac{\partial T_H}{\partial y} - k_E \frac{\partial T_E}{\partial y} + p_I V_I c_{PH} T_E \right) = -w_I h^0.$$

Here we have used the definitions

$$\frac{D_y}{Dt} \equiv \frac{\partial}{\partial t} + v \frac{\partial}{\partial y}, \quad c_{PH} \equiv \frac{5K}{2m_A}, \quad k_H = k_I + k_A. \quad (3.19)$$

Together with the transport properties, the ionization rates, and the elastic energy exchange given in Section II, and the gas law indicated in Eq. (3.11), these equations determine the variables v , α , T_H and T_E . (Remember that only two of the three energy equations (3.18c, d, e) are needed.)

An expression for the total energy flux normal to the wall can be obtained from Eq. (3.12), with the addition of the flux of ionization energy. This expression is

$$-q = k_H \frac{\partial T_H}{\partial y} + k_E \frac{\partial T_E}{\partial y} - p_I V_I (c_{PH} T_E + h^0). \quad (3.18f)$$

Boundary Conditions

At the outside edge of the layer, we take all conditions to be known; with subscript e denoting these conditions, we have

$$T_H = T_{He}, \quad T_E = T_{Ee}, \quad \alpha = \alpha_e. \quad (3.20)$$

We will assume the external plasma to be in temperature equilibrium so that $T_{He} = T_{Ee} = T_e$.

The wall boundary conditions involve consideration of the presence of

the plasma sheath. The continuum approach is used up to the edge of the sheath. In the sheath the usual molecular description is used. The inner boundary conditions for the solution of Eqs. (3.18) are obtained by a matching of the continuum and molecular descriptions.

As boundary conditions we will use the continuity of ion mass flux and electron energy flux at the interface between the continuum and molecular descriptions. The former requires

$$\frac{m_I n_I c_I}{4} = -\rho_I V_I, \quad (3.21)$$

while the latter condition can be obtained from Eqs. (3.12) and the considerations of Section II. Electrons which get to the wall enter the sheath with average energy $2kT_E + |e\phi|$ and the flux of electrons which reach the wall is $(n_E \bar{c}_E/4) \exp(-|e\phi|/kT_E)$. Thus, continuity of electron energy flux requires

$$\rho_E \frac{\partial T_E}{\partial y} - \rho_I V_I c_{PI} T_E = (2kT_E + |e\phi|) \frac{n_E \bar{c}_E}{4} \exp\left(-\frac{|e\phi|}{kT_E}\right). \quad (3.22)$$

The potential ϕ is determined by Eq. (2.11). The electron flux is replaced by the continuum ion flux by using Eqs. (3.21) and (2.10) with $J = 0$.

The third boundary condition determines T_H at the wall. The atoms, unaffected by the sheath voltage, are in good contact with the wall at the wall temperature. The ions are in temperature equilibrium with the atoms, except possibly within a few sheath thicknesses of the wall. However, the continuum description ends where this equilibrium no longer holds, so as a boundary condition as the continuum equations, we take the heavy temperature as the wall temperature:

$$T_H = T_w. \quad (3.23)$$

The three boundary conditions (3.21)-(3.23) are applied at $y = 0$, and complete the specification of the problem.

Method of Solution

The problem, as so far formulated, requires solution of partial differential equations. If only the case of temperature equilibrium is considered, the chemically frozen and equilibrium cases permit similarity solutions in which the introduction of the independent variable

$$\eta \equiv \left(\frac{c_{pe}}{2 + \rho_e k_e} \right)^{1/2} \int_0^y \rho dy \quad (3.24 a)$$

permits a reduction to ordinary differential equations. When either temperature or chemical nonequilibrium is considered, such an exact reduction is not possible. If η is introduced into Eqs. (3.18) and (3.20)-(3.23), t appears explicitly in the E_{el} and w_I terms, as well as in Eq. (3.21). However, a "local similarity" type of approach is possible, with t treated as a parameter, and the normalized dependent variables

$$\theta_H(\eta) \equiv T_H/T_e, \quad \theta_E(\eta) \equiv T_E/T_e, \quad \Delta(\eta) \equiv \alpha/\alpha_e \quad (3.24b)$$

taken as functions of η only. This approach assumes the rate of change with t to be small, and all the important time dependence to be contained in the explicit appearance in the source terms and boundary conditions. Its great advantage is the fact that only one independent variable is involved, so only ordinary differential equations must be solved.

This "local similarity" approach is the one which has been used in the few calculations so far made.

IV. RESULTS AND DISCUSSION

To the date of writing only two solutions have been obtained with the local similarity approach outlined here. A third solution for temperature equilibrium is available from a previous investigation.¹⁹ The present calculations represent one external condition, for one time, with two different values of the elastic collision rate constant C_{el} , and with no gas phase recombination. The conditions used are those behind a reflected shock for an initial argon pressure of 1 mm Hg, and an incident shock velocity of 6 mm/ μ sec. These conditions are

$$T_e = 15,828^\circ\text{K}, \quad p = 2.516 \text{ atm}, \quad \alpha_e = 0.558,$$

$$T_w = 300^\circ\text{K}, \quad t = 1 \mu\text{sec}$$

The two values of C_{el} so far used are

$$C_{el} = 0 \text{ and } C_{el} = 0.005$$

For the determination of c_I in Eq. (2.11) both the limiting values discussed in Section II, namely

$$c_I = \sqrt{8KT_w/\pi m_I} \quad \text{and} \quad c_I = \sqrt{8K(2\pi T_e)/\pi m_I},$$

were used for each value of C_{el} , since the number of collisions in the sheath is 2 or 3, which is between the many-collision and zero-collision limits. Numerical difficulties have so far prevented the solutions for higher values of C_{el} , although the appropriate value is $C_{el} = 1$.⁵

The profiles for the cases $C_{el} = 0$ and $C_{el} = \infty$ (temperature equilibrium) are shown in Fig. 1. These two cases should represent limits of weak and strong elastic energy exchange for frozen chemistry. Only one set of profiles is shown for $C_{el} = 0$, since the difference between using the two limiting values of c_I is not discernible on the scale of the figure. The profiles for $C_{el} = 0.005$ also differ only slightly from those for $C_{el} = 0$.

We see that for the case of no energy exchange by elastic collisions the electron temperature at the edge of the sheath is nearly half the external gas temperature, and remains nearly constant through the rather thin diffusion and heavy temperature layer, and then rises gradually in a layer some four times as thick. This high electron temperature at the sheath edge is brought about by the insulation of the electrons from the wall by the negative sheath potential. Only a small fraction of one percent of the electrons reaching the sheath edge get through the sheath to the wall. However, the fraction ionized at the sheath edge is quite high (3×10^{-3} to 3×10^{-4}) because the enhancement of ambipolar diffusion by the high

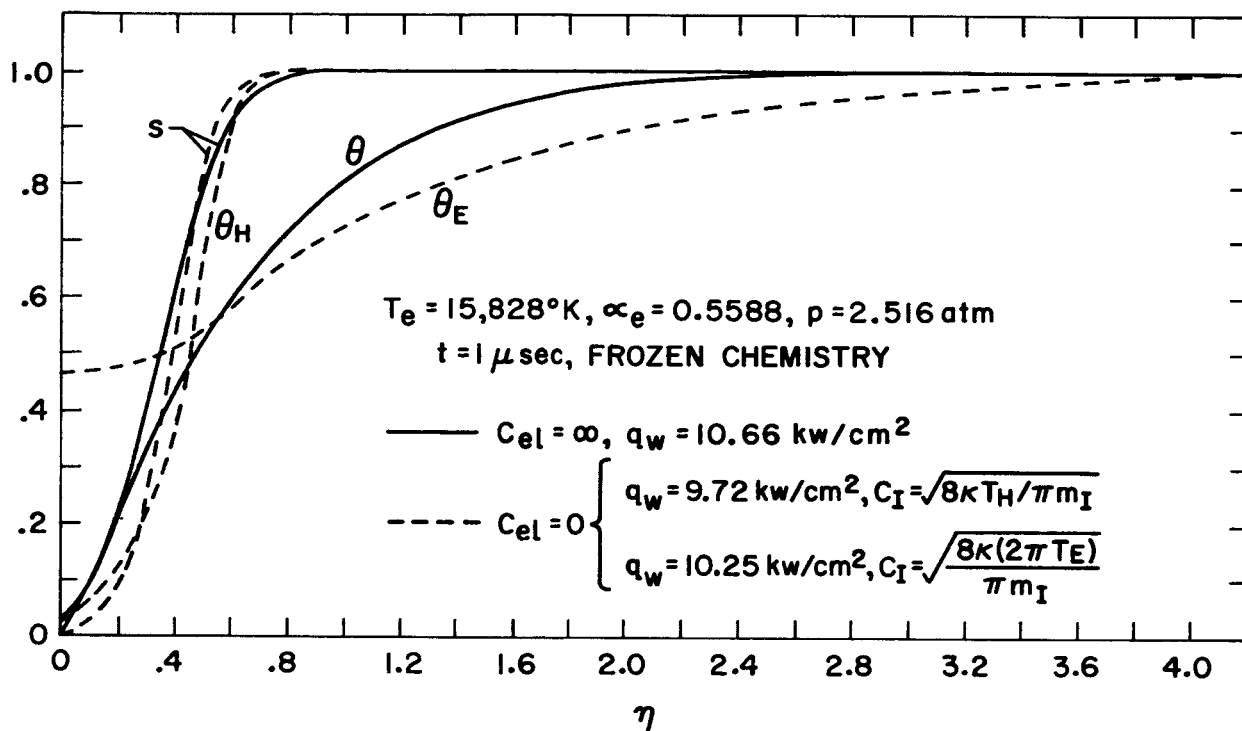


Fig. 1 Profiles of normalized temperature and fraction ionized in terms of the similarity variable η for frozen chemistry.

electron temperature brings in a large flux of electron-ion pairs. So even the small fraction of electrons getting through carries enough energy to reduce the electron temperature below its external value.

The heat transfer to the wall can be found by calculating the heat transfer to the sheath edge, since the sheath is so thin that all this energy gets to the wall. For this purpose we use Eq. (3.18f) applied at $\eta = 0$. The results are given in Fig. 1 for both values of c_I . We see that the case of no energy exchange by elastic collisions yields a slightly smaller heat transfer rate than the case with temperature equilibration. This is probably a result of the thin heavy temperature layer, which in turn is caused by the uncoupling of the electrons from the heavies. The differences caused by using the two limiting values of c_I is also small.

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